description of its principal characters. A few of the important species in each genus are quite fully described and in many cases illustrated. These are followed by a further enumeration of a number of other species with their hosts and localities, the species in many cases for Britain and the United States being indicated.

The book is very fully illustrated, a very large number of the illustrations being new, either from the pencil of the author or from excellent photographs. As foot notes, there are very copious references to works even in cases where space would not permit of a discussion of their contents.

Neither the author nor the translator pretends to completeness, but modestly offer excuses for faults which under the conditions could not be well avoided. These can well be overlooked in view of the great amount of information contained in the volume which will prove to be a very useful adjunct to reference works on parasitic fungi. When a new German edition shall be called for the author promises to thoroughly revise it and expresses the wish that those who have in the past sent him copies of their investigations continue to do so in order that he may make this edition as complete as possible.

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RECENT BOOKS ON QUATERNIONS.

1. *Theorie der Quaternionen.* VON DR. P. MOLENBROEK. Leiden, E. J. Brill. 1891. Pp. vii+284.
2. *Anwendung der Quaternionen auf die Geometrie.* By the same author. 1893. Pp. xv+257.
3. *The Outlines of Quaternions.* By. LIEUT.-COL. H. W. L. HIME. London, Longmans & Co. 1894. Pp. 190.
4. *A Primer of Quaternions.* By A. S. HATHAWAY. New York, Macmillan & Co. 1896. Pp. x+113.
5. *Utility of Quaternions in Physics.* By A. MCAULAY. Macmillan & Co. 1893. Pp. xiv +107.

The above books are all contributions to the literature of the Quaternion side of space-analysis. The first, by Dr. Molenbroek, is a carefully written exposition of Hamilton's theory; the author, if he does not examine the correspondence of the theory with exact science and established analysis, at least presents it so as to be internally consistent. For instance, he explains the fundamental rule $ij=k$ as meaning that a quadrant round the axis j followed by a quadrant round the axis i is equivalent to a quadrant round the axis k. Consistently with this, he explains the rule $i^2=-1$ as meaning that a quadrant round the axis i followed by a quadrant round the same axis is equivalent to a reversal. The treatise, however, does not go deep enough; for the subject of quaternion logarithms and exponentials is embraced in a 9-page appendix, and what is there given is the well-known theory of coplanar exponentials. It is only when diplanar exponentials are handled that problems can be attacked which are insoluble, or at least not readily solved by the ordinary methods of analysis. Dr. Molenbroek introduces an indefinite use of $\sqrt{-1}$ to signify a quadrant round some axis perpendicular to a given line. There are reasons for believing that in space-analysis $\sqrt{-1}$ is scalar in its nature, and that it distinguishes the hyperbolic angle from the circular angle. Anyhow, that is one definite meaning.

The third book, by Col. Hime, presents a very dim and imperfect outline, which it would be well for the beginner to avoid. By perusing it he may get his ideas confused, not only of analysis, but of mechanics; for example, at p. 33 the terms 'version,' 'torsion,' 'rotation,' 'twist,' are all used as synonymous. This is, at least, awkward, for one of the first things which a student of quaternions must do is to distinguish between the trigonometrical composition of angles and the mechanical composition of rotations. The author explains the rule $ij=k$ by saying that j and k each signify a unit vector, but i signifies a quadrantal versor which turns j into k. But he fails to observe that this explanation cannot apply to the complementary rule $i^2=-1$, for a quadrantal versor i operating on a unit vector i would leave it i. Chapter Tenth is devoted to the 'Interpretation of Quaternion Expressions;' thus for nine chapters the reader is supposed to be dealing with symbolical expressions. Would it not be

better if the real meaning of each expression were clear from the beginning?

The fourth book, by Professor Hathaway, presents a much better introduction to the method, and the student who masters it will find that he has acquired some real knowledge, not merely additional dexterity in formal manipulations. The exposition, as a matter of logic and of truth, is not all that can be desired, for it is based partly on formal laws, partly on mechanical truths. For example, the principle that the addition of vectors is associative is made to depend on an arbitrary definition of the equality of vectors, but the same principle for the product of quaternions is rested upon the composition of rotations of a rigid body.

The fifth book, by Mr. McAulay, has a different purpose from that of the others. It is an essay, not an introduction or a treatise, and the aim of the essay is to make good the following statements: First, that Quaternions are in such a stage of development as already to justify the practically complete banishment of Cartesian geometry from physical questions of a general nature; and second, that Quaternions will in physics produce many new results that cannot be produced by the rival and older theory. In the essay the author applies the quaternion analysis to the theories of elastic solids, electricity and magnetism and hydrodynamics. *It is almost wholly a translation into quaternion notation of known results; the author has,* however, endeavored to advance each of the theories mentioned in at least one direction.

It is evident that the utility of a method is best proved not by any essay, but by its extensive and fruitful use. How does it come about that the method of quaternions is so far from general and accepted use that it is still the subject of debate, misunderstanding and even ridicule? Not a few mathematicians agree with the opinion expressed by a German mathematician, that it is an aberration of the human intellect. The answer to the above question I believe to be as follows:

In the books before us, and, indeed, in all the works by members of the old school, it is admitted, even proclaimed, that the Hamiltonian analysis is a rival of the Cartesian analysis. Mr. McAulay talks of it as a new plant, independent of the old tree of analysis; and in their letter to SCIENCE proposing an international association Dr. Molenbroek and Mr. Kimura invited mathematicians to leave the old domain of Cartesian analysis. Now, when one who has been trained in the Cartesian analysis approaches the new method he finds that the notation is strange and the conventions contradictory of those to which he has been accustomed; consequently, he concludes, as David did about Saul's armor, that it is better in actual warfare to rely on a familiar weapon than on one which may be superior but is unproved.

What is the true relation of space-analysis to the Cartesian analysis? The quaternionist makes them rivals; there is the blunder. Space-analysis can be presented so as not to contradict or rival the Cartesian analysis, but, on the contrary, be consistent with and supplementary to it. The relation of the former to the latter is like that of algebra to arithmetic. Algebra is universal arithmetic; so space-analysis is universal Cartesian analysis; that is, it considers the properties of vectors which are independent of coordinates. Many theorems are readily proved by algebra which it would be difficult, if not impossible, to prove by arithmetic; similarly, many theorems can be readily proved by space-analysis which it is *difficult, if not impossible, to prove by means of coordinates. If we wish numerical results,* coordinates must be introduced, just as, if we wish numerical results, numbers must be introduced into the formula furnished by algebra.

Some writers express the opinion that agreement about notation is all that is required in order to render space-analysis generally accepted. But it appears to me that the difficulty is more deep-seated; the fundamental principles need to be discussed, and no notation can be adequate and lasting which is not built on the simplest and truest principles. I may mention briefly some points of principle which have to be settled.

It is unscientific to base the analysis partly on formal laws, partly on physical principles. By not distinguishing between simultaneous and successive addition Hamilton failed to discover the true generalization for space of the

exponential theorem. I have demonstrated that in space $e^p \times e^q = e^{p+q}$, and the demonstration shows conclusively that the Hamiltonian ideas about the addition of vectors require to be revised. Although I have asked quaternionists to point out any error in the demonstration, no error has been pointed out.

The Hamiltonian principle that a unit-vector may be identified with a quadrantal versor requires to be modified. The conception of a line does not involve the idea of an angle, whereas the conception of an angle involves the idea of two lines. The question reduces to the following: Can a line be conceived apart from an initial line? The answer appears to be yes, for Hamilton did not succeed in his endeavors to extend algebra to space until he abandoned the idea of an initial line and considered all three axes as equally real. The vector and the versor are complementary ideas, and just as a vector is expressed in terms of rectangular coordinates which are in their nature vectors, so a versor is expressed in terms of rectangular quadrantal coordinates which are in their nature versors.

On the other hand, a vector cannot take the place of the versor. To ignore the versor and more generally the quaternion is the mistake made by writers who confine space-analysis to vector-analysis, which is merely a branch. The very name vector-analysis implies a restricted view of space-analysis. The versor is the proper idea in spherical trigonometrical analysis, and in a modified form expresses the rotation of a rigid body. It leads up to higher ideas which express elliptic and hyperbolic angles and the motion of a body which is not rigid.

In mathematical analysis the product of two quantities having the same direction is positive, while that of two quantities having opposite directions is negative; consequently the square of a quantity is always positive. Consistent with this the reciprocal of a negative quantity is the negative of the reciprocal. Now, are all the quantities considered in algebra or the Cartesian analysis scalar quantities, or are they in some cases partial vectors? If in any case they are partial vectors (that is, component of a vector) then, in order to be consistent, the square of a vector in space must be positive and the reciprocal of a vector have the same direction as the vector.

The order of writing of the terms of a sum or the factors of a product should conform, as far as possible, to the order followed in mathematical analysis. There the natural order of writing is followed, from left to right, and, as in a determinant, from top to bottom. But in books on Quaternions, for example, Hathaway's Primer, p. 49, we have the Hebrew order of writing. This abnormal order of writing was adopted from the idea that a product of quaternions supposed an operand and that the operand ought to be on the right. As a matter of fact, in the expression for the rotation of a versor the operator is written both before and behind.

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SCIENTIFIC JOURNALS.

JOURNAL OF GEOLOGY, APRIL–MAY.

PROFESSOR CHAMBERLIN continues his glacial studies in Greenland, giving a description of the Bowdoin glacier. This is a tongue of the great inland ice-cap which descends from the north into the head of Bowdoin Bay. On the west it is confluent with the Tuktoo and Sun glaciers. The Bowdoin glacier has a length of six or eight, and in its lower part a breadth of about two miles. It has a descent of 2,000 to 3,000 feet, and is notably crevassed. It discharges icebergs of considerable dimensions, the discharge varying greatly with the season. The west side does not present the usual vertical scarp, and this is thought to be due to the fact that the ground which should act as a reflecting plane is covered by protuberances from the Tuktoo glacier. The stratification and basal loading of the ice is much the same as in the glaciers previously described, though the débris does not rise so high. The bowlders were usually more rounded, and this rounding is of such a nature as to imply very considerable wear. This considerable rounding, the small amount of débris and its low position in the ice are especially significant in view of the fact that the Bowdoin is one of the larger tongues of the great icecap.

Dr. Henry Washington describes the Rocca Monfino region in the fourth of his Italian